Announcements

1) HW due Thursday

2) No 3rd quiz.

3) Exam next Thursday

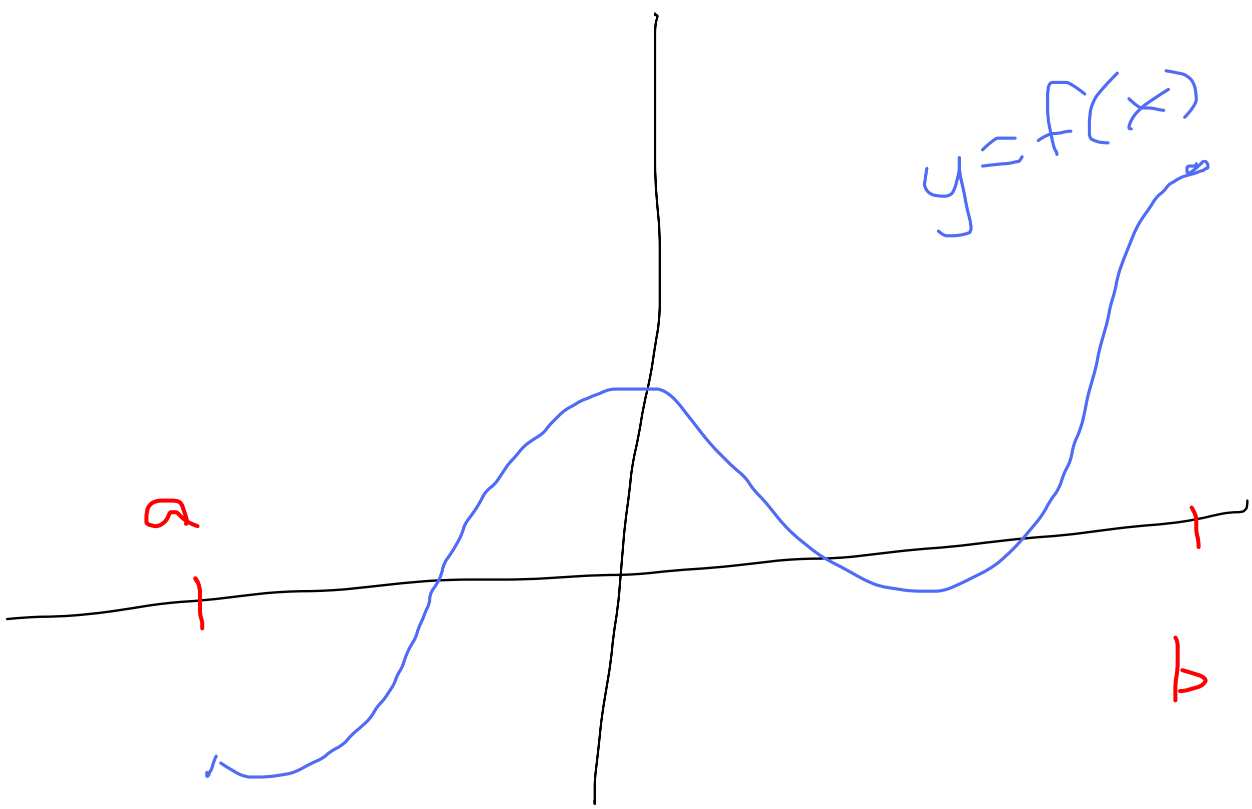
One more fact about Integrals

$$\int_a^a f(x)\,dx = 0$$

for any function f.

Left and Right Sums

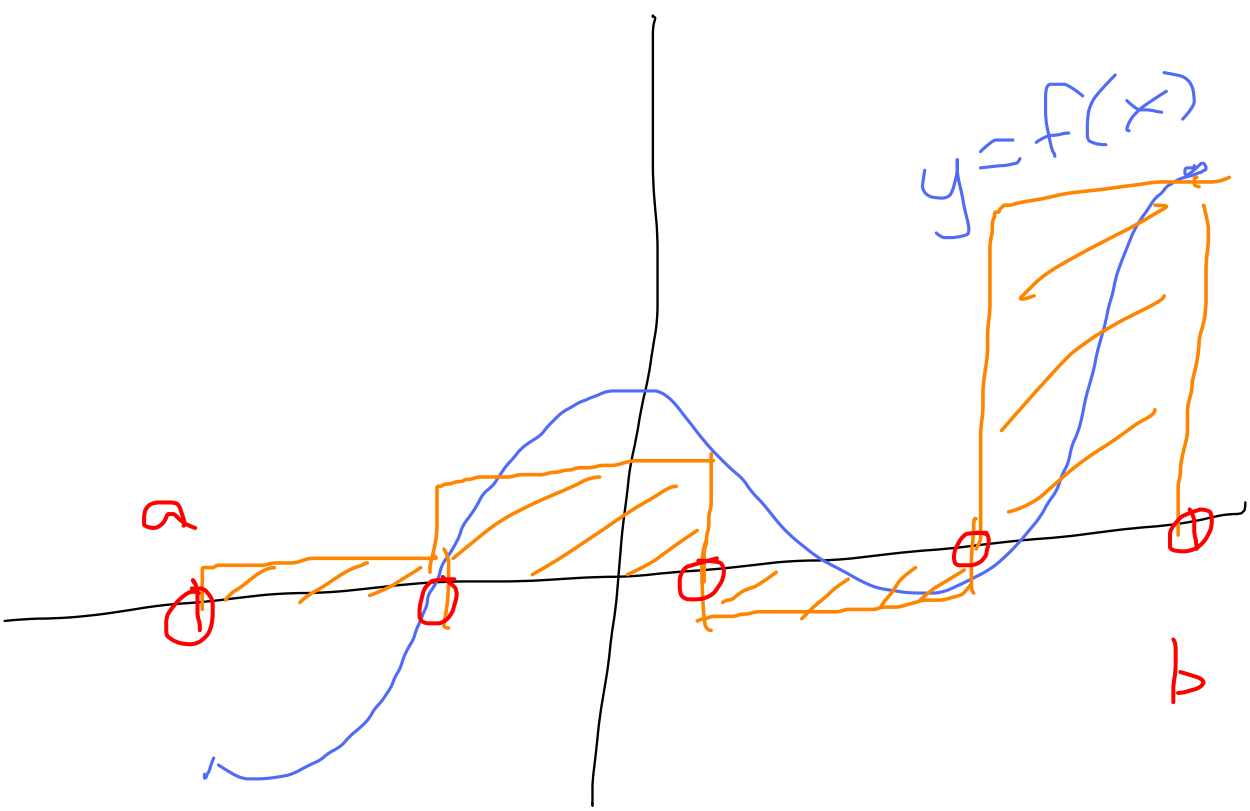
Given f on $[a, b]$



to find the integral, we used

$$\int_a^b f(x)\,dx$$

$$= \lim_{n \to \infty} \frac{b-a}{n} \left(f\left(a + \frac{b-a}{n}\right) + f\left(a + \frac{2(b-a)}{n}\right) + \cdots + f\left(a + \frac{(n-1)(b-a)}{n}\right) + f(b) \right)$$

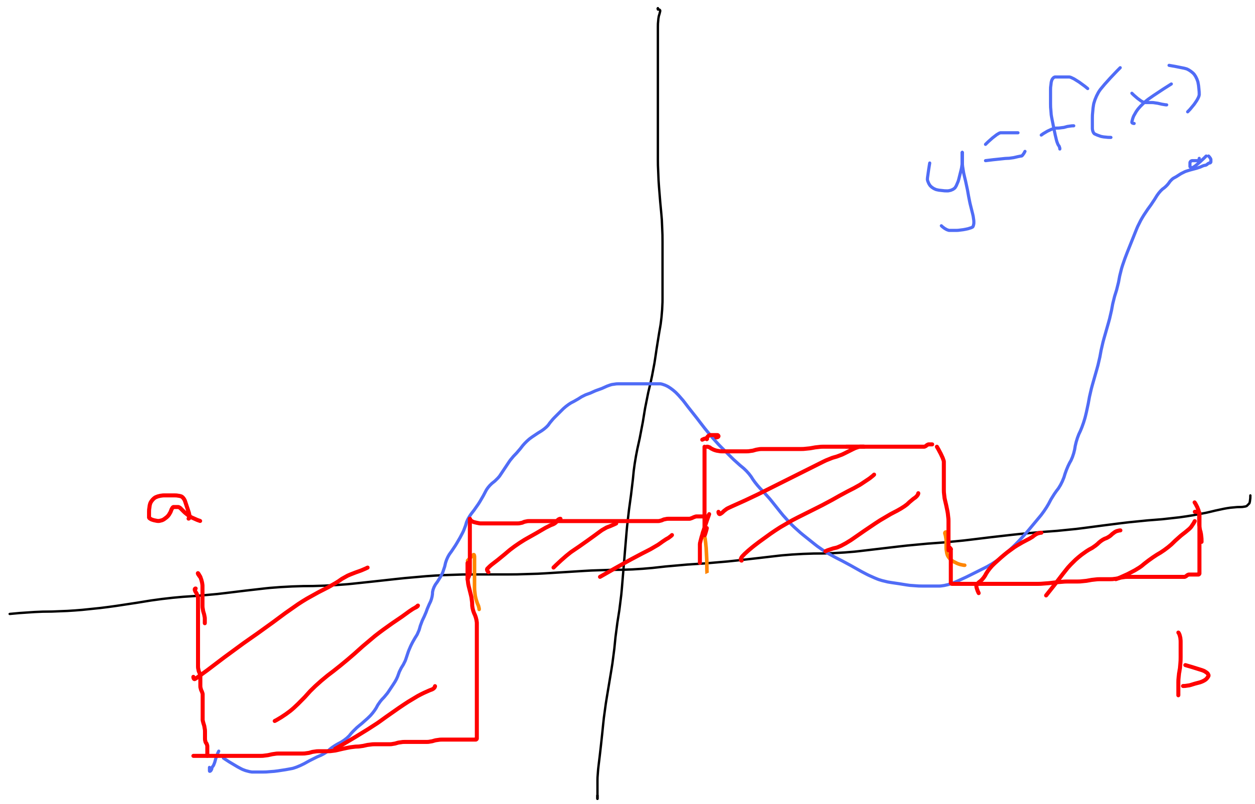


This is a right-hand sum. We could have just as easily used a left-hand sum

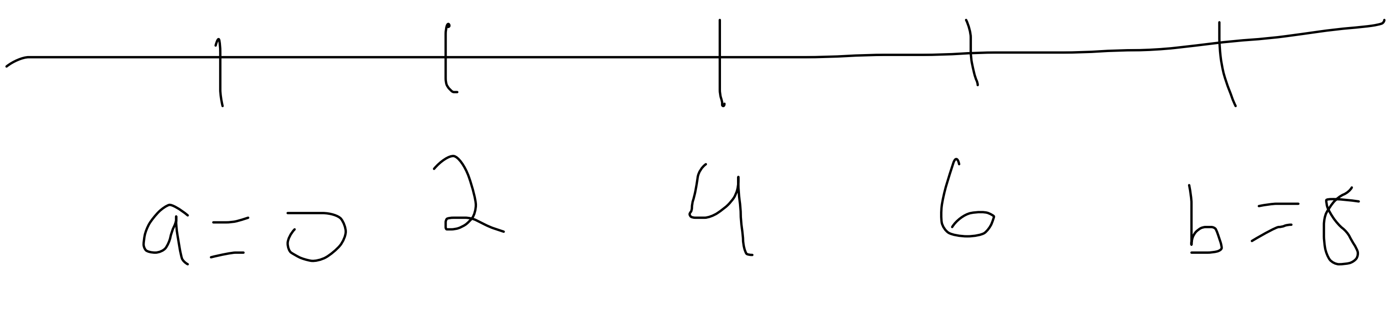
$$\int_a^b f(x)\,dx$$

$$= \lim_{n\to\infty} \frac{b-a}{n}\left(f(a) + f\left(a + \frac{(b-a)}{n}\right) + \cdots + f\left(a + \frac{(n-1)(b-a)}{n}\right)\right)$$

Picture



Example 1: Compute the left-hand and right-hand sums with four subdivisions for $f(x) = \sqrt{1+x^3}$ on the interval $[0, 8]$.



Right hand sum

$$\underbrace{\frac{8-0}{4}}_{\frac{b-a}{n}}\left(f(2)+f(4)+f(6)+f(8)\right)$$

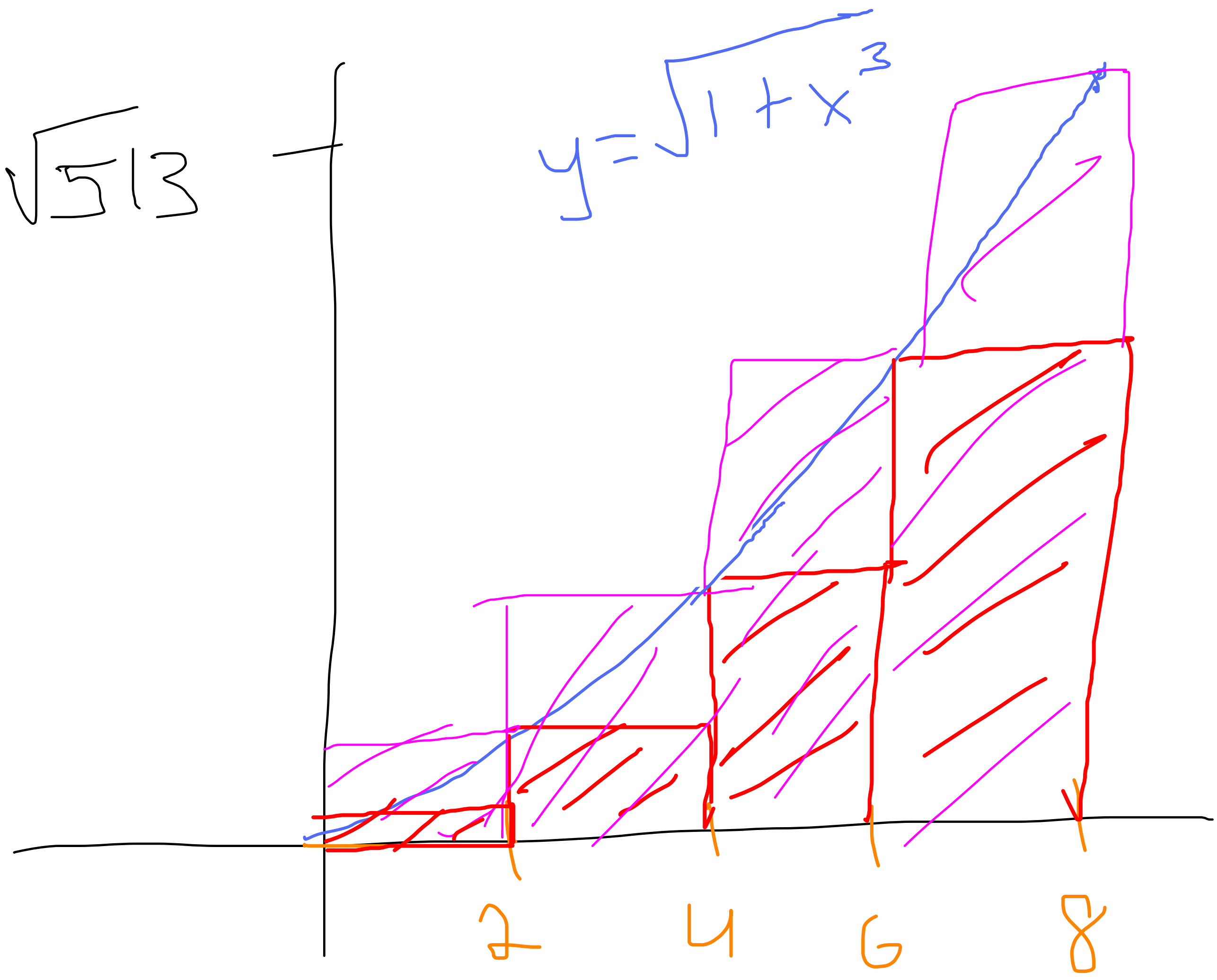
$$= 2\cdot\left(3+\sqrt{65}+\sqrt{217}+\sqrt{513}\right)$$

Left hand sum

$$\frac{8-0}{4}\left(f(0)+f(2)+f(4)+f(6)\right)$$

$$= 2\left(1+3+\sqrt{65}+\sqrt{217}\right)$$

Picture



red = left-hand

purple = right-hand

Fundamental Theorem of Calculus

Section 4.3

(2 parts)

Let f be a continuous function on $[a, b]$

Note: If f is continuous,

$\int_a^b f(x)\,dx$ always exists.

Part One

If $a \le x \le b$, define a function

$$\boxed{g(x) = \int_a^x f(t)\,dt}$$

Then g is continuous on $[a,b]$, differentiable on (a,b), and for all x with $a \le x \le b$,

$$\boxed{g'(x) = f(x)}$$

Part Two

If h is any antiderivative of f,

$$\boxed{\int_a^b f(x)\,dx = h(b) - h(a)}.$$

Example 2:

$$\int_0^{\pi} \sin(x)\,dx$$

Use fundamental theorem, part 2.

An antiderivative for $f(x) = \sin(x)$ is $h(x) = -\cos(x)$.

Then

$$\int_0^{\pi} \sin(x)\,dx = -\cos(\pi) + \cos(0)$$

$$= \boxed{2}$$

Example 3:

$$\int_0^1 \left(x^{1/3} + \sec^2\left(\frac{\pi x}{4}\right)\right) dx$$

$$= \int_0^1 x^{1/3}\,dx + \int_0^1 \sec^2\left(\frac{\pi x}{4}\right) dx$$

An antiderivative for

$f(x) = x^{1/3}$ is

$h(x) = \dfrac{3x^{4/3}}{4}$

An antiderivative for

$g(x) = \sec^2\left(\frac{\pi x}{4}\right)$ is

$k(x) = \tan\left(\frac{\pi x}{4}\right) \cdot \frac{4}{\pi}$ chain rule

So the integral is

$h(1) - h(0) + k(1) - k(0)$

$= \frac{3}{4} + \underbrace{\tan\left(\frac{\pi}{4}\right)}_{=1} \frac{4}{\pi} + 0 \; \frac{4}{\pi}$

$= \boxed{\frac{3}{4} + \frac{4}{\pi}}$

Example 4: $f(x) = \sqrt{1-x^2}$

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx$$

Antiderivative of $\sqrt{1-x^2} = ?$

$$\cancel{h(x) = \frac{(1-x^2)^{3/2}}{3/2} \cdot \frac{1}{2x}}$$

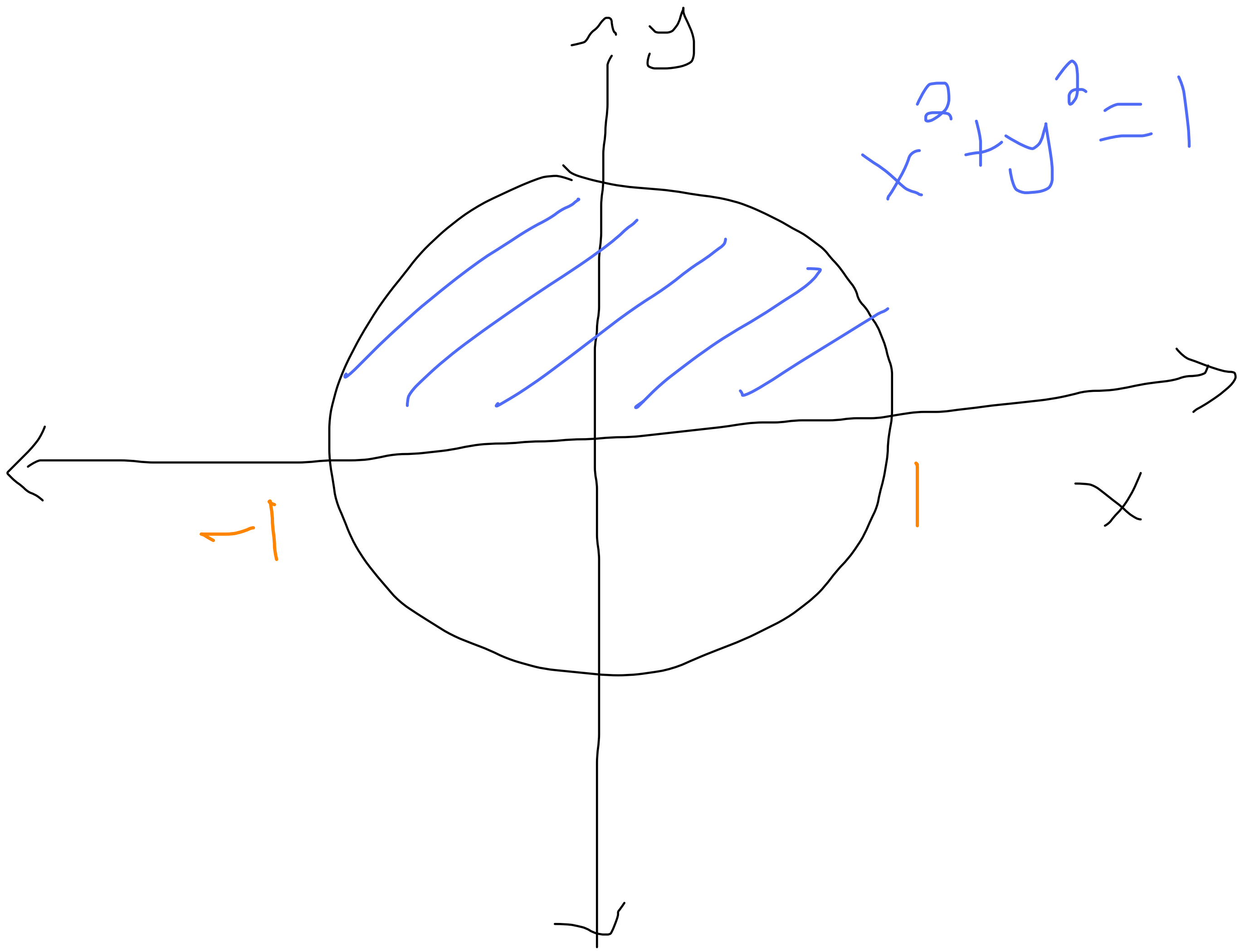
For right now, we don't know an antiderivative.

Use geometry

$$y = f(x) = \sqrt{1-x^2}$$

$$y^2 = 1 - x^2$$

$x^2 + y^2 = 1$ – a circle of radius 1 and center $(0,0)$



$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{1}{2}(\text{area of circle})$$

$$= \frac{1}{2}\pi = \boxed{\frac{\pi}{2}}$$

Why is part 2 true?

If f is continuous, we know from part 1 that

$$g(x) = \int_a^x f(t)\,dt,$$

then $g'(x) = f(x)$.

If h is any other antiderivative of f,

$$h(x) = g(x) + C$$

for some constant

C since g is an

antiderivative for f,

and any two antiderivatives

differ by a constant.

$$g(b) = \int_a^b f(t)\,dt$$

$$g(a) = \int_a^a f(t)\,dt = 0$$

Then $g(b) - g(a) = \int_a^b f(t)\,dt.$

If h is any antiderivative of f,

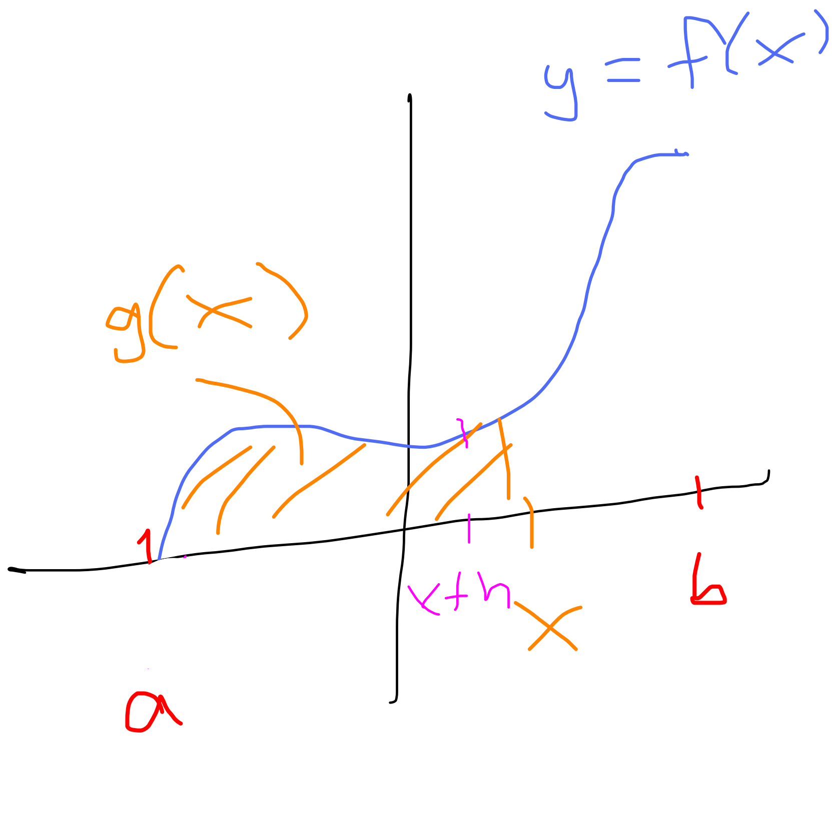
$$h(b) - h(a) = (g(b) + c) - (g(a) + c)$$
$$= g(b) + \cancel{c} - g(a) - \cancel{c}$$
$$= g(b) - g(a)$$
$$= \int_a^b f(t)\,dt \checkmark$$

Why is part 1 true?

Consider our function

$$g(x) = \int_a^x f(t)\,dt$$

Picture.



By definition,

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt}{h}$$

(properties of integrals)

$$= \lim_{h \to 0} \frac{\left(\cancel{\int_a^{x} f(t)\,dt} + \int_x^{x+h} f(t)\,dt\right) - \cancel{\int_a^{x} f(t)\,dt}}{h}$$

$$= \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

Let M_h = maximum of f on $[x, x+h]$

m_h = minimum of f on $[x, x+h]$

By properties of integrals,

$$\boxed{m_h \leq f(t) \leq M_h}$$

so

$$\int_x^{x+h} m_h\,dt \leq \int_x^{x+h} f(t)\,dt \leq \int_x^{x+h} M_h\,dt$$

$$\| \qquad\qquad\qquad\qquad\qquad\qquad$$

$$h \cdot m_h \leq \int_x^{x+h} f(t)\,dt \leq h \cdot M_h$$

$$h \cdot m_h \leq \int_x^{x+h} f(t)\,dt \leq h \cdot M_h$$

Divide by h

$$m_h \leq \frac{\int_x^{x+h} f(t)\,dt}{h} \leq M_h$$

take limit as $h \to 0$

$m_h, M_h \to f(x)$ so

$$f(x) \leq \underbrace{\lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}}_{g'(x)} \leq f(x)$$

By squeeze theorem,

$$g'(x) = f(x)$$

Done!